Journal of Engineering Physics and Thermophysics, Vol. 81, No. 4, 2008

ANALYSIS OF THREE-DIMENSIONAL WAVE MOTIONS IN A CUBICALLY ANISOTROPIC THERMOELASTIC MEDIUM WITH ALLOWANCE FOR THE RELAXATION TIME OF THERMAL DISTURBANCES

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Formulas for the coordinates of points of a cubically anisotropic thermoelectric medium, which determine the geometry of wave fronts excited by a lumped disturbance source, have been obtained. The three-dimensional fronts of a quasilongitudinal modified elastic wave, a modified thermal wave, and quasitransverse elastic waves propagating in lead have been constructed.

Introduction. The regularities of propagation of plane waves and discontinuity surfaces in isotropic and anisotropic media whose thermal properties are described by a generalized (hyperbolic) heat-conduction law have been the focus of quite numerous works [1–3]. In particular, two-dimensional wave motions in a cubically anisotropic medium and a transversally isotropic one have qualitatively and quantitatively been analyzed in [3] and the wave fronts of thermoelastic waves have been visualized. Below, we present results of investigation of the three-dimensional fronts of elastic and thermal waves propagating, in a cubically anisotropic medium, from a lumped source with allowance for the interrelation between the thermal and mechanical fields.

Coordinates of Wave-Front Points. Following [3], we represent a resolving system of differential equations for thermoelastic anisotropic materials of a cubic symmetry system in the form

$$\left(A_4\Delta + (A_1 - A_2 - 2A_4)\,\partial_i^2\right)u_i + (A_2 + A_4)\,\partial_i \sum_{k=1}^{3} \partial_k u_k = \rho\partial_t^2 u_i + \beta\partial_i T\,, \tag{1}$$

$$c_\varepsilon \partial_t T + T_0\beta \sum_{k=1}^{3} \partial_t\partial_k u_k = -\sum_{k=1}^{3} \partial_k q_k\,, \quad \tau\partial_t q_i + q_i = -\lambda\partial_i T\,, \quad i = \overline{1, 3}\,.$$

We specify the initial conditions to system (1) on the surface $z(x_1, x_2, x_3, t) = 0$ and pass to new variables z, z_1, z_2, and z_3 according to the following scheme:

$$g = z\,(x_1, x_2, x_3, t)\,, \quad g_i = z_i\,(x_1, x_2, x_3, t)\,, \quad i = \overline{1, 3}\,.$$

We substitute expressions for the derivatives of first and second orders with respect to the variables x_1, x_2, x_3, and t expressed by the variables g_1, g_2, g_3, and g into the system of equations (1) and set the determinant composed of the coefficients of the partial derivatives $\dfrac{\partial^2 u_i}{\partial g^2}$ and $\dfrac{\partial T}{\partial g}$, $\dfrac{\partial q_i}{\partial g}$, $i = \overline{1, 3}$, equal to zero. As a result we obtain the characteristic equation

$$p_0^2\left(\frac{k_0 p_0^8}{c_1^8} + \frac{k_1 p_0^6}{c_1^6} + \frac{k_2 p_0^4}{c_1^4} + \frac{k_3 p_0^2}{c_1^2} + k_4\right) = 0\,. \tag{2}$$

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The coefficients of Eq. (2) have the form

$$k_0 = -n_*\,, \quad k_1 = \left(1 + n_*\,(1 + 2b + \varepsilon)\right)\delta_1\,, \tag{3}$$

$$k_2 = -\left(1 + b^2 n_* + 2b\,(1 + n_* + \varepsilon n_*)\right)\left(\delta_1^2 - 2\delta_2\right)$$
$$-\left(2 + 2b^2 n_* + b\,(4 - 2\,(a-1)\,n_*) - (a-1)\,n_*\,(1 + a + 2\varepsilon)\right)\delta_2\,, \tag{4}$$

$$k_3 = b\left(2 + b\,(1 + n_* + \varepsilon n_*)\right)\left(\delta_1^3 - 3\delta_1\delta_2 + 3\delta_3\right) + \left(4b^3 n_* + 6\,(a-1)\,(an_* + \varepsilon n_* - 1) + 6b^2\,(an_* + \varepsilon n_* + 1)\right.$$
$$\left. -\,(a-1)\,(3 + n_* + 3\varepsilon n_* - 2a^2 n_* + a\,(3 + n_* - 3\varepsilon n_*))\right)\delta_3 + \left(1 - a^2\,(1 + bn_*) + b^2\,(3 + n_* - \varepsilon n_*)\right.$$
$$\left. -\,2ab\,(1 + bn_* + \varepsilon n_*) + b\,(4 + n_* + 2\varepsilon n_*)\right)\left(\delta_1\delta_2 - 3\delta_3\right), \tag{5}$$

$$k_4 = -\,b^2\left(\delta_1^4 - 4\delta_1^2\delta_2 + 2\delta_2^2 + 4\delta_1\delta_3\right) + (a-1)\,(1 + a + 2b)\,b$$
$$\times\left(\delta_1^2\delta_2 - 2\delta_2^2 - \delta_1\delta_3\right) + 2b\left(a^2 - 1 - b + 2ab\right)\left(\delta_2^2 - 2\delta_1\delta_3\right)$$
$$-\left(1 + 2a^3 + 2b + 2ab\,(b-3) + 2b^2\,(1 + 2b) + a^2\,(4b - 3)\right)\delta_1\delta_3\,. \tag{6}$$

The characteristic equation (2) yields the existence of a stationary discontinuity surface ($p_0 = 0$), three modified elastic waves whose propagation is influenced by the temperature field, and a modified thermal wave whose propagation is accompanied by elastic deformations. To find the coordinates of the medium's points approached by the energy of wave disturbance from a point source by the instant of time t we express p_0 from Eq. (2)

$$p_0^{(n)} = \sqrt{\frac{1}{2}\left(\delta^{(n)}\sqrt{z_0} - (-1)^n\sqrt{-2\delta^{(n)} q - \sqrt{z_0}\,(2p + z_0)}\right)}\,, \tag{7}$$

where $\delta^{(n)} = \delta_{n1} + \delta_{n2} - \delta_{n3} - \delta_{n4}$, $\delta_{nj} = 1$ if $n = j$, and $\delta_{nj} = 0$ if $n \neq j$; the superscript points to the type of wave: $n = 1$ is the quasilongitudinal modified elastic wave, $n = 2$ and $n = 3$ are the quasitransverse modified elastic waves, and $n = 4$ is the thermal modified wave. The remaining coefficients in formulas (7) have the form

$$z_0 = \sqrt{2\left(\sqrt{-\frac{P}{3}}\cos(\Lambda) - \frac{p}{3}\right)}\,, \quad \Lambda = \frac{1}{3}\arccos\left(-\frac{Q}{2}\sqrt{-\left(\frac{3}{P}\right)^3}\right),$$
$$P = -\frac{p^2 + 12r}{3}\,, \quad Q = \frac{8pr}{3} - \frac{2p^3}{27} - q^2\,, \tag{8}$$
$$p = \frac{1}{k_0}\left(k_2 - \frac{3k_1^2}{8}\right), \quad q = \frac{1}{k_0}\left(\frac{k_1^3}{8} - \frac{k_1 k_2}{2} + k_3\right), \quad r = \frac{1}{k_0}\left(\frac{k_1^2 k_2}{16} - \frac{k_1 k_3}{4} - \frac{3k_1^4}{256} + k_4\right).$$

Differentiating relations (7) with respect to the parameters p_i, after simple transformations we obtain expressions for the dimensionless coordinates $\left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$ of points of the three-dimensional fronts of thermoelastic waves [4] propagating in a cubically anisotropic medium ($i = \overline{1, 3}$ and $n = \overline{1, 4}$):

$$\frac{x_i^{(n)}}{c_1 t} = \frac{\dfrac{\delta^{(n)} z_0^{(i)}}{2\sqrt{\hat{z}_0}} + (-1)^n \dfrac{4\hat{q}_i \delta^{(n)} \sqrt{\hat{z}_0} + z_0^{(i)} (2\hat{p} + \hat{z}_0) + 2\sqrt{\hat{z}_0}\left(2\hat{p}_i + z_0^{(i)}\right)}{4\sqrt{\hat{z}_0}\sqrt{-2\hat{q}\delta^{(n)} - \sqrt{\hat{z}_0}\,(2\hat{p} + \hat{z}_0)}}}{\sqrt{2\left(\delta^{(n)}\sqrt{\hat{z}_0} - (-1)^n \sqrt{-2\delta^{(n)}\hat{q} - \sqrt{\hat{z}_0}\,(2\hat{p} + \hat{z}_0)}\right)}}. \tag{9}$$

Here we introduce the following notation:

$$z_0^{(i)} = \frac{3\sqrt{2}}{\sqrt{\sqrt{-3\hat{P}}\cos(\hat{\Lambda}) - \hat{p}}}\left(\frac{\hat{P}_i \cos(\hat{\Lambda})}{2\sqrt{-3\hat{P}}} - \sin(\hat{\Lambda})\,\hat{\Lambda}_i \sqrt{-\frac{\hat{P}}{3}} - \frac{\hat{p}_i}{3}\right),$$

$$\hat{\Lambda}_i = \frac{\sqrt{3}\left(3\hat{Q}\hat{P}_i - 2\hat{P}\hat{Q}_i\right)}{2\hat{P}^4\sqrt{-4\hat{P}^3 - 27\hat{Q}^2}}, \quad \hat{P}_i = -\frac{2(\hat{p}\,\hat{p}_i + 6\hat{r}_i)}{3}, \quad \hat{Q}_i = \frac{8}{3}(\hat{p}\,\hat{r}_i + \hat{p}_i\hat{r}) - \frac{2\hat{p}^2\hat{p}_i}{9} - 2\hat{q}\hat{q}_i,$$

$$\hat{p}_i = \frac{1}{k_0}\left(\hat{k}_{2i} - \frac{3\hat{k}_1\hat{k}_{1i}}{4}\right), \quad \hat{q}_i = \frac{1}{k_0}\left(\frac{3\hat{k}_1^2\hat{k}_{1i}}{8} - \frac{\hat{k}_{1i}\hat{k}_2 + \hat{k}_1\hat{k}_{2i}}{2} + \hat{k}_{3i}\right),$$

$$\hat{r}_i = \frac{1}{k_0}\left(\frac{2\hat{k}_1\hat{k}_{1i}\hat{k}_2 + \hat{k}_1^2\hat{k}_{2i}}{16} - \frac{\hat{k}_{1i}\hat{k}_3 + \hat{k}_1\hat{k}_{3i}}{4} - \frac{3\hat{k}_1^3\hat{k}_{1i}}{64} + \hat{k}_{4i}\right),$$

$$k_{1i} = 2\left(1 + n_*(1 + 2b + \varepsilon)\right)n_i, \quad k_{2i} = -4\left(1 + b^2 n_* + 2b(1 + n_* + \varepsilon_* n_*)\right)n_i^3$$

$$-2\left(2 + 2b^2 n_* + b(4 - 2(a-1)n_*) - (a-1)n_*(1 + a + 2\varepsilon)\right)n_i\left(\hat{\delta}_1 - n_i^2\right),$$

$$k_{3i} = 6b\left(2 + b(1 + n_* + \varepsilon n_*)\right)n_i^5 + \left(4b^3 n_* + 6(a-1)(an_* + \varepsilon n_* - 1) + 6b^2(an_* + \varepsilon n_* + 1)\right.$$

$$\left.-2(a-1)\left(3 + n_* + 3\varepsilon n_* - 2a^2 n_* + a(3 + n_* - 3\varepsilon n_*)\right)\right)n_i\left(\hat{\delta}_2 - n_i^2\left(\hat{\delta}_1 - n_i^2\right)\right)$$

$$+\left(1 - a^2(1 + bn_*) + b^2(3 + n_* - \varepsilon n_*) - 2ab(1 + bn_* + \varepsilon n_*)\right.$$

$$\left.+2bn_i(4 + n_* + 2\varepsilon n_*)\right)\left(\hat{\delta}_1\left(\hat{\delta}_1 + 2n_i^2\right) - 3n_i^4 - 2\hat{\delta}_2\right),$$

$$k_{4i} = -8b^2 n_i^7 + 2bn_i(a-1)(1 + a + 2b)\left(3n_i^4\left(\hat{\delta}_1 - n_i^2\right) + 3\hat{\delta}_3 + \hat{\delta}_1^3 - 3\hat{\delta}_1\hat{\delta}_2 - n_i^6\right)$$

$$+8bn_i^3(a^2 - 1 - b + 2ab)\left(\hat{\delta}_1^2 - 2\hat{\delta}_2 - n_i^4\right) - 2n_i\left(\hat{\delta}_3 + \hat{\delta}_1\left(\hat{\delta}_2 - n_i^2\left(\hat{\delta}_1 - n_i^2\right)\right)\right)$$

$$\times\left(1 + 2a^3 + 2b + 2ab(b-3) + 2b^2(1 + 2b) + a^2(4b - 3)\right).$$

Expressions for $\hat{k}_i$ are obtained by replacement of the coefficients δ_i in formulas (3)–(6) by $\hat{\delta}_i$, where $\hat{\delta}_1 = n_1^2 + n_2^2 + n_3^2$, $\hat{\delta}_2 = n_1^2 n_2^2 + n_3^2 n_2^2 + n_1^2 n_3^2$, and $\hat{\delta}_3 = n_1^2 n_2^2 n_3^2$ ($n_i = \cos\alpha_i$ are the direction cosines of the normal slope to the characteristic surface and α_i is the angle between the normal to the characteristic surface and the coordinate axis x_i). Other quantities necessary for determining the coordinates $\left(x_1^{(n)}, x_2^{(n)}, x_3^{(n)}\right)$ will be obtained by replacing the coefficients z_0, Λ, P, Q, p, q, and r in relations (8) by $\hat{z}_0$, $\hat{\Lambda}$, $\hat{P}$, $\hat{Q}$, $\hat{p}$, $\hat{q}$, and $\hat{r}$ respectively.

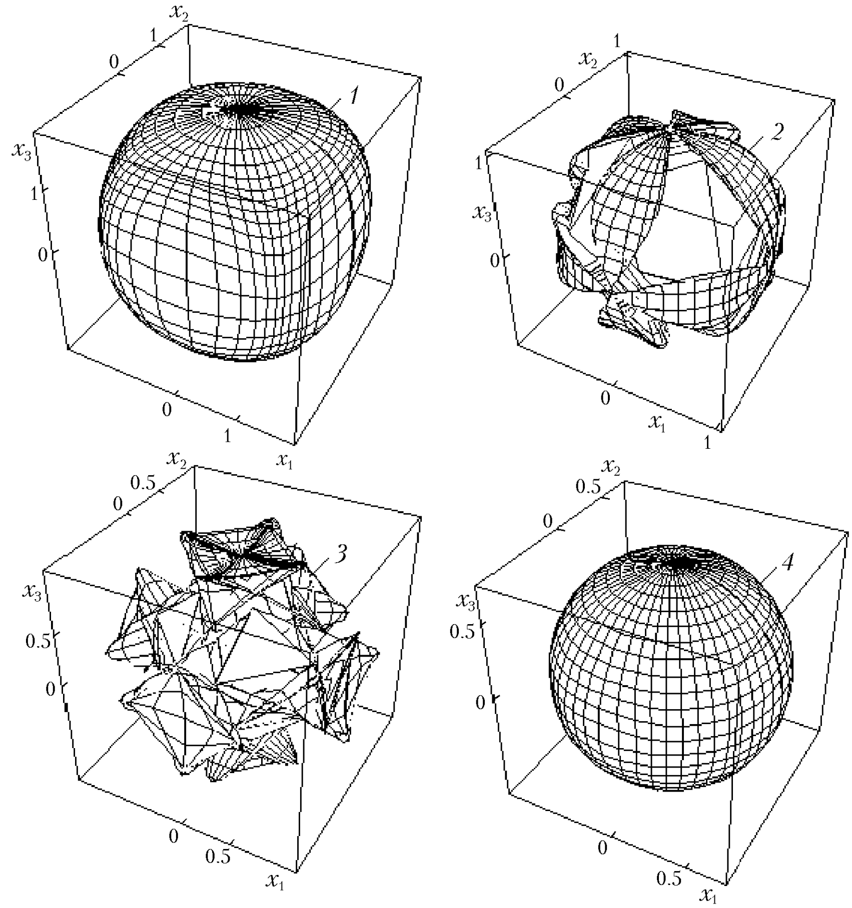

Fig. 1. Three-dimensional fronts of thermoelastic waves propagating in lead: 1) quasilongitudinal modified elastic wave; 2 and 3) quasitransverse elastic waves (the geometry of the fronts is determined by the coordinates $x^{(2)}/c_1t$ and $x_i^{(3)}/c_1t$ respectively); 4) modified thermal wave.

Wave Fronts. We apply formulas (9) to construction of dimensionless three-dimensional fronts of modified elastic and thermal waves propagating in a cubically anisotropic medium. Thus, Fig. 1 shows the position of wave surfaces in lead, which corresponds to the instant of time $t = 1$ sec. In the computations, we use the following physicomechanical constants: $A_1 = 46.6$, $A_2 = 39.2$, and $A_4 = 14.4$ GPa, $\rho = 11{,}342$ kg/m^3, $\alpha_t = 28.35 \cdot 10^{-6}$ 1/K, $\lambda = 35$ W/(m·K), $T_0 = 293$ K, $c_\varepsilon = 1458$ kJ/(K·m^3), and $\tau = 1 \cdot 10^{-11}$ sec ($a = 0.84$, $b = 0.31$, $\varepsilon = 0.609$, and $n = 1.72$; the numerical data have been taken from [5–7]). To find the absolute values of the coordinates of the wave-front points we must multiply the values along the coordinate axes by a value numerically equal to c_1t. We note that analogous three-dimensional and two-dimensional fronts of thermoelastic and elastic waves are observed in cubically anisotropic materials: silver, gold, brass, nickel, and many other metals belonging to the cubic symmetry system and forming the first group of cubically anisotropic materials [8].

For quantitative evaluation of the influence of the effect of interrelation between the mechanical and temperature fields on the propagation of elastic and thermal waves we consider the sections of wave surfaces by the planes passing through the coordinate axis x_3. A comparative analysis of the corresponding two-dimensional wave fronts for $\varepsilon \neq 0$ and $\varepsilon = 0$ for different cubically anisotropic materials shows that the most significant difference of the modified waves from pure elastic and thermal waves is observed in the coordinate planes $x_i = 0$, $i = \overline{1, 3}$. The sections of the fronts of the quasilongitudinal elastic and thermal waves propagating in lead, by the coordinate plane $x_1 = 0$, with allowance for the interrelation between the thermal and mechanical fields and without it, are presented in Fig. 2 (numerical data required for calculation have been given above.)

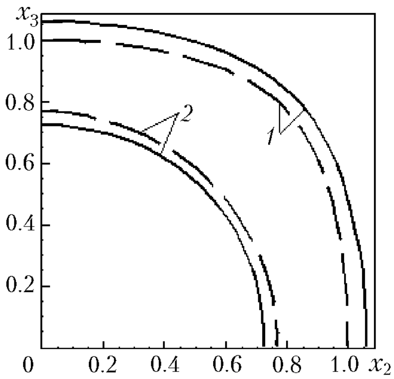

Fig. 2. Two-dimensional fronts of waves propagating in the coordinate plane $x_1 = 0$ of a cubically anisotropic material (the solid and dashed curves have been constructed with allowance for the influence of the temperature field and without it respectively): 1) quasilongitudinal modified elastic wave; 2) modified thermal wave.

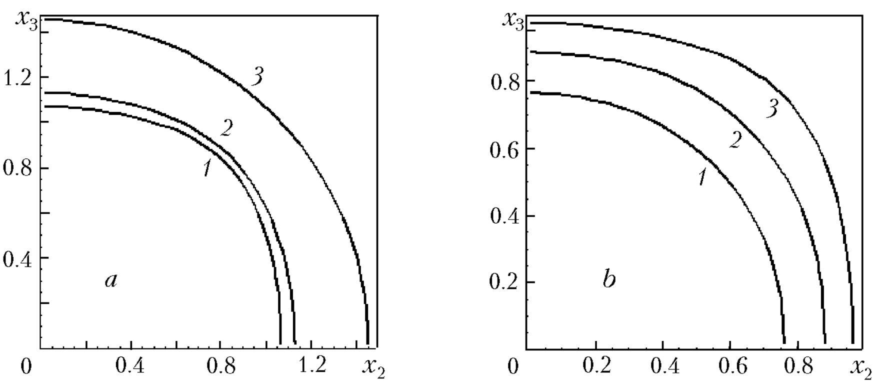

Fig. 3. Two-dimensional fronts of the quasilongitudinal modified elastic (a) and modified thermal waves (b) propagating in the plane $x_1 = 0$ of a cubically anisotropic material: 1) $n_* = 1.5$, 2) 1, and 3) 0.5.

From Fig. 2, it is clear that allowance for the interrelation between the temperature and deformation fields leads to an increase in the velocity of propagation of the quasilongitudinal elastic wave and to a decrease in the velocity of the thermal wave. The greatest change in the velocity is observed near the coordinate axes; in particular, for lead, the increase in the velocity of the quasilongitudinal modified elastic wave amounts to $\approx$7% compared to the velocity of the corresponding pure elastic wave, whereas the decrease in the velocity is nearly 6% compared to the pure thermal wave. The influence of the temperature field on the propagation of the quasitransverse elastic waves in cubically anisotropic materials has not been found in any of the planes (coordinate or noncoordinate).

Influence of the Relaxation Time on the Wave Front. The value of the relaxation time of wave disturbances for most cubically anisotropic media has neither been determined nor been determined for cases where the material was at low temperatures [3]. Therefore, results of investigation of the influence of the relaxation time τ on the propagation of elastic waves is of practical and theoretical interest, since these results can be used to identify τ values for different cubically anisotropic media. Figure 3 gives the sections of the three-dimensional fronts of the quasilongitudinal modified elastic wave and the modified thermal wave propagating in lead by the coordinate plane $x_1 = 0$; the sections have been constructed for a characteristic number of vibrations n_* of 1.5, 1.0, and 0.5, which corresponds to τ values of 8.77, 5.85, and 2.92 psec.

From Fig. 3, it is clear that the relaxation time of thermal disturbances exerts a significant influence on the propagation of elastic-wave fronts; the radial velocities of propagation of two modified waves grow with decrease in τ. In particular, as the n_* values decrease from 1.5 to 0.5, the radial velocity of the quasilongitudinal modified elastic wave grows by nearly 30%, whereas the radial velocity of the modified thermal wave increases by 20%. In the case where $\tau \to 0$ the radial velocities of propagation of the modified elastic and thermal waves tend to infinity.

Conclusions. The results obtained enable one to visualize wave motions occurring in thermoelastic cubically anisotropic media due to the action of a nonstationary point source. Three-dimensional wave fronts and their sections can be used in the mechanics of rigid body and physical acoustics in conducting natural experiments on determination of physicomechanical constants and correct interpretation of experimental data.

NOTATION

A_1, A_2, and A_4, elastic constants, Pa; $a = A_2/A_4$; $b = A_4/A_1$; $c_1 = \sqrt{A_1/\rho}$, m/sec; c_ε, specific heat at constant deformation, J/(K·m^3); $n_i = \cos \alpha_i$, direction cosines of the normal slope to the characteristic surface; $n_* = \tau\omega_*$, characteristic number of vibrations; $p_i = \partial z/\partial x_i$; $p_0 = \partial z/\partial t$; q_i, components of the vector of surface heat-flux density, W/m^2; T, change in the absolute temperature, K; T_0, initial temperature, K; t, time, sec; u_i, components of the displacement vector; m; α_i, angle between the normal to the characteristic surface and the coordinate axis x_i; α_t, coefficient of linear thermal expansion, 1/K; $\beta = (A_1 + 2A_2)\alpha_t$; Δ, Laplace operator; $\delta_1 = p_1^2 + p_2^2 + p_3^2$; $\delta_2 = p_1^2p_2^2 + p_3^2p_2^2 + p_1^2p_3^2$; $\delta_3 = p_1^2p_2^2p_3^2$; $\hat{\delta}_1 = n_1^2 + n_2^2 + n_3^2$, $\hat{\delta}_2 = n_1^2n_2^2 + n_3^2n_2^2 + n_1^2n_3^2$, $\hat{\delta}_3 = n_1^2n_2^2n_3^2$; $\varepsilon = T_0\beta^2/(A_1c_\varepsilon)$, dimensionless connectivity coefficient; λ, thermal conductivity, W/(m·K); ρ, density, kg/m^3; τ, relaxation time of thermal disturbances, sec; $\omega_* = c_\varepsilon A_1/(\lambda\rho)$, characteristic quantity having the dimensions of frequency, 1/sec. Subscript: t, thermal.

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